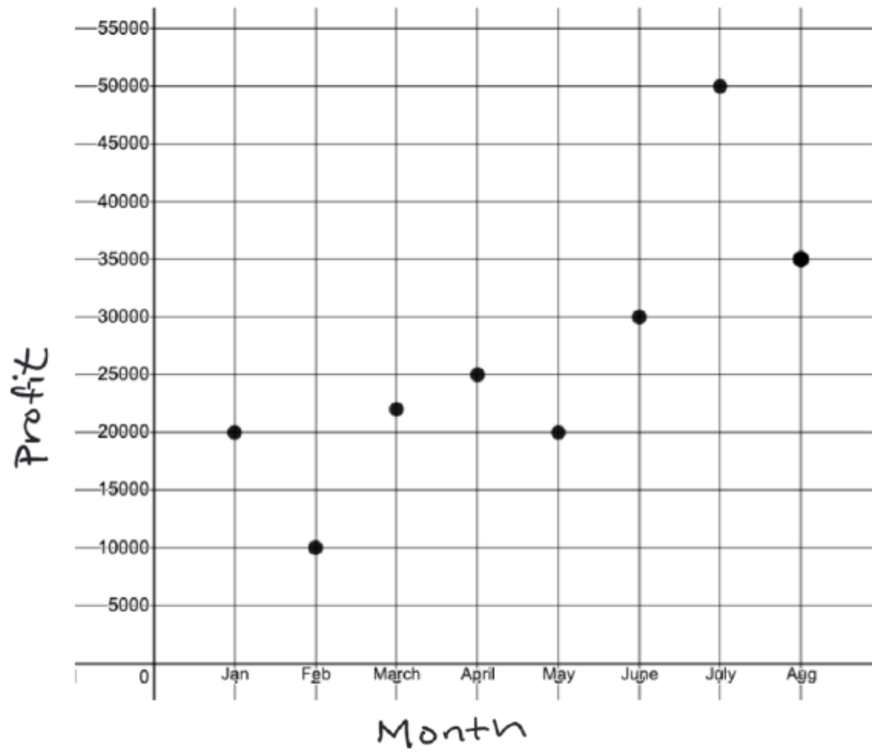
3.1 Introduction to Extrema

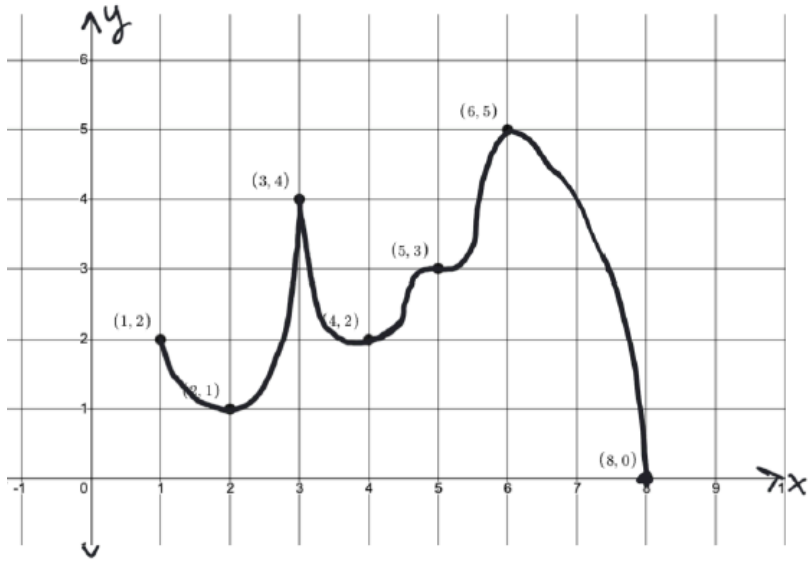
The following graph depicts the profit for a company from through August. If this was your company, what would be of notable interst?



Absolute Extrema vs Local Extrema

> **1 Definition** Let c be a number in the domain D of a function f. Then $f(c)$ is the
> - **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
> - **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

An absolute maximum or minimum is sometimes called a **global** maximum or minimum. The maximum and minimum values of f are called **extreme values** of f.

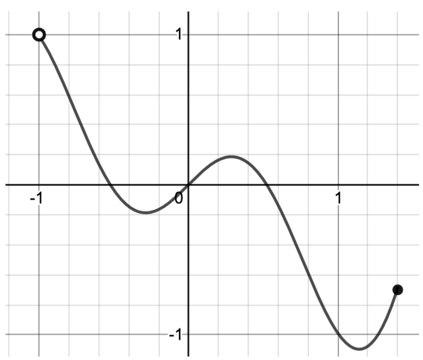


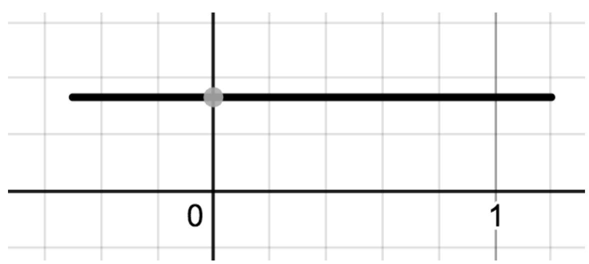
> **2 Definition** The number $f(c)$ is a
> - **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
> - **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

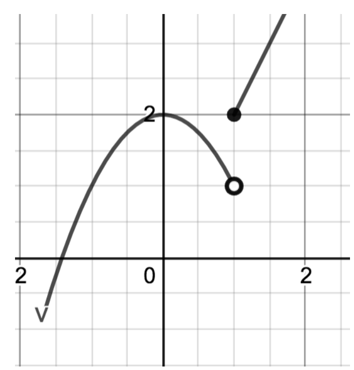
Clarification: " x is near a" :____ so local extrema cannot occur at _________

If there is an absolute extremum at c then there is also an local extremum at x=c unless ______________________________

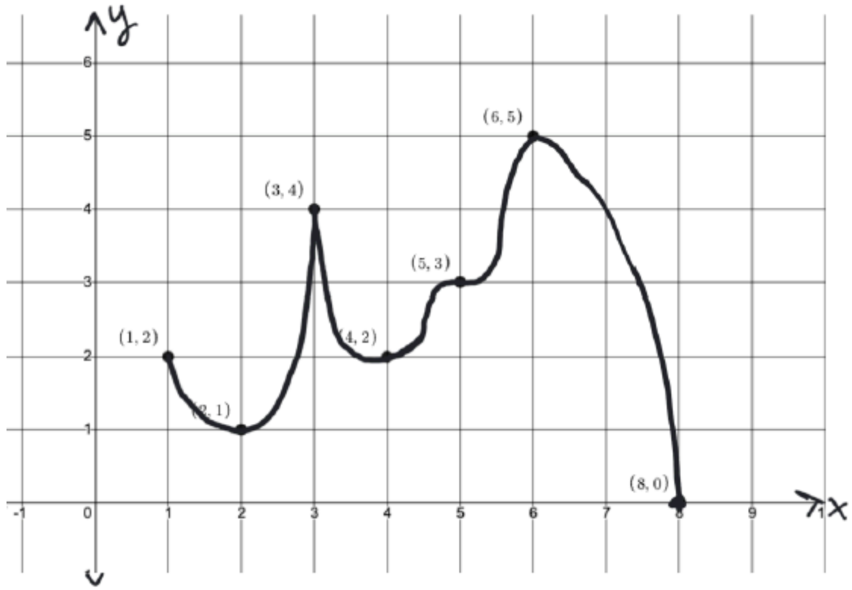
A little tricker: Identify extrema for each graph







Vocabulary Details and Observations:



The absolute max is ___________. It occurs when x= ___________ at the point ________

The absolute min is ___________. It occurs when x= ___________ at the point ________

There are local minimums at the points:__________________________

There are local maximums at the points: _________________________

There are horizontal tangents at x=________________________

f(x) fails to be differentiable at x= __________________________

If there is a local extremum for f at x=c then ______________________________ (These are called critical numbers)

If $f'(c)=0$ or f is not differentiable at c (i.e. if c is a critical number) then f(x) _______________ have a local extrema at x=c. (see x=5)

Critical Numbers: A critical number of a function is a number c in the domain of f such that $f'(c)=0$ or $f'(c)$ does not exist:

<u>Find the critical numbers:</u>

$f(x)=x^3+x^2-x$ $f(x)=\sqrt{x}$ $f(x)=\frac{1}{x^2}$

Recall, critical numbers give us a list of possibilities ______________________________.

So finding critical numbers is ______________________________step to finding extrema.

What's next? How do we know if these critical numbers actually yield extrema?

Start with a special case, then in 3.3 and 3.7 we will broaden our search.

Special case: f(x) continuous on a closed interval [a,b]

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

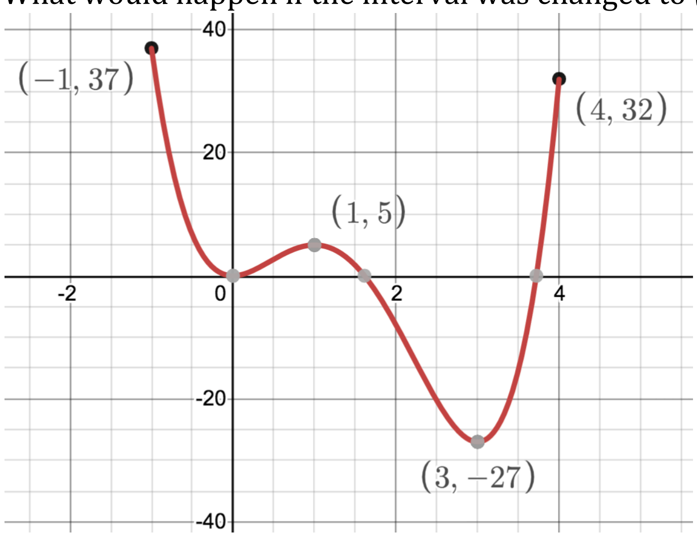
What did we observe earlier about where absolute extrema can occur?

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example: Find the absolute extrema of $f(x) = 3x^4 - 16x^3 + 18x^2$ on [-1,4]

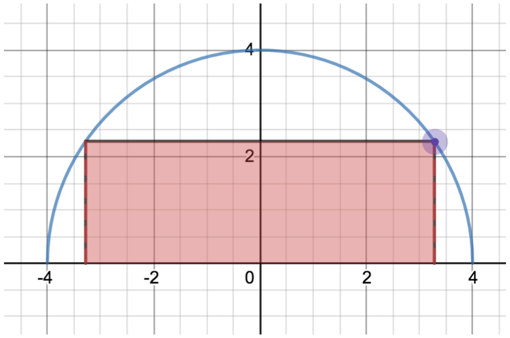
What would happen if the interval was changed to (-1,4] ?



What about local extrema?

Applied problems: 3.7i

Example: What are the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 4?
(see 5A page for Desmos animation: https://www.desmos.com/calculator/lfiles3gvj)



From last time…

Local extrema

Local extrema can only occur at ______________________________

Critical numbers occur for x values in the domain of f(x) such that _____________________ or _____________________

Critical numbers give a list of _________________________x values for where local extrema _______________________ occur.

It is possible to have a critical number that __________________________ lead to an extremum. (For example, $f(x)=x^3$ at $x=0$)

A function may or may not have any local extrema.

Local extrema cannot occur at _______________________ of the domain.

So out first step is to find critical numbers. Then what?.....3.3

Absolute extrema

If a function has absolute extrema, they occur at a ______________________ or at an __________________ of the domain.

A function may not have any absolute extrema.

So out first step is to find critical numbers. Then what?.....3.7ii

In the special case where f(x) is continuous on a closed interval [a,b],the Extreme Value Theorem tells us ________________

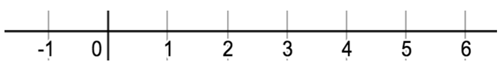
So <u>in this special case</u>, we need only find critical numbers and ____
(See: 3.1 Closed Interval Method)

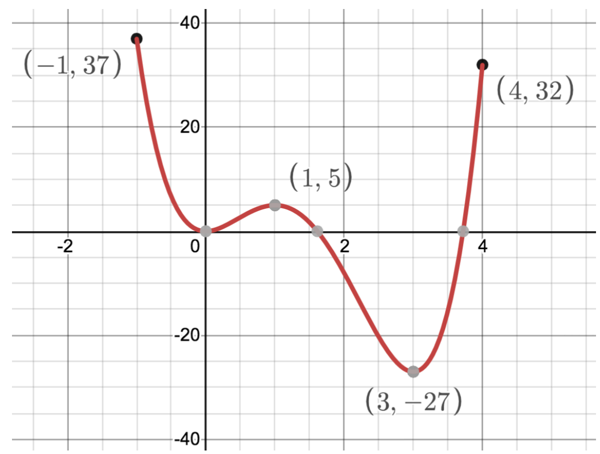
3.3 Local Extrema and What Derivatives Tell us about a Graph.

Local Extrema – What does $f'(x)$ tell us about $f(x)$?

Last time we looked at the absolute extrema of $f(x) = 3x^4 - 16x^3 + 18x^2$ on [-1,4]

We found $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x-3)(x-1)$. When x=0,1,3, $f'(x) = 0$, so x=0,1,3 are our critical numbers. This means our function _______________ have local extrema at these values. If there ARE any local extrema, they can only happen at these values. But how do we know if these critical numbers yield local extrema. Consider what the sign of $f'(x)$ tells us.





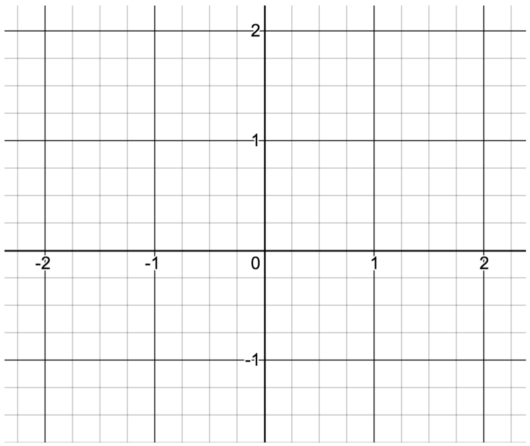
The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

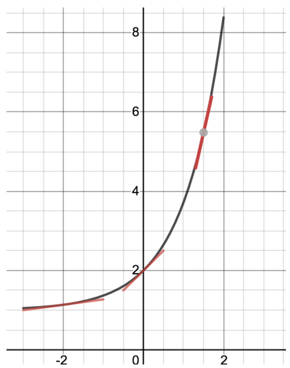
Example: Find local extrema for $f(x) = x^3 + x^2 - x$. Where is f increasing? Decreasing? Use this information to sketch a graph.



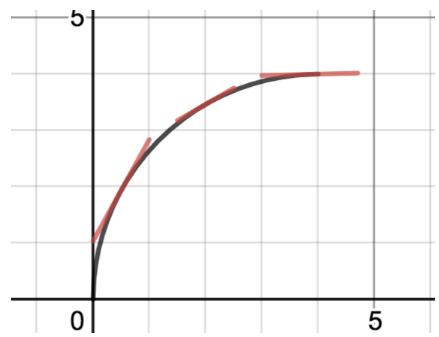
Example: Find local extrema for $f(x) = 5x^{2/3} - 2x^{5/3}$.

Concavity

Consider two functions, both increasing on the given interval. Remember that if a function is increasing, then____



Concave UP



Concave DOWN

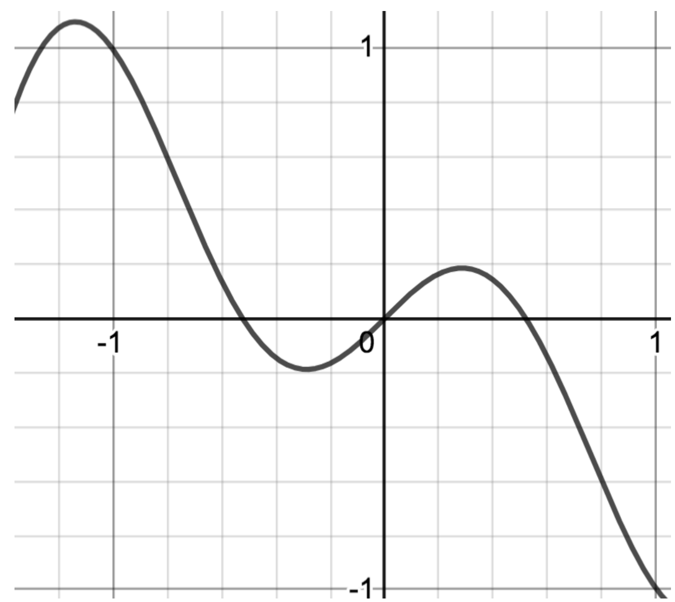
Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.



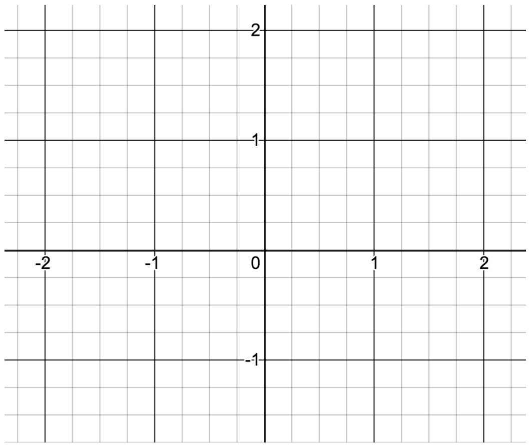
More on local extrema: Second Derivative Test

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Revisit the earlier example and discuss concavity and inflection points. $f(x) = x^3 + x^2 - x$



Example: Given $f(x) = 5x^{2/3} - 2x^{5/3}$ find $f'(x)$, $f''(x)$ and use them to graph the function.

What can we learn about the graph from $f(x)$?

Domain

Behavior near domain issues/Discontinuities/Vertical Asymptotes

X intercepts

Y intercepts

Symmetry

End behavior

What can we learn about the graph from $f'(x)$?

Horizontal Tangents/Vertical Tangents/Cusps

Increasing

Decreasing

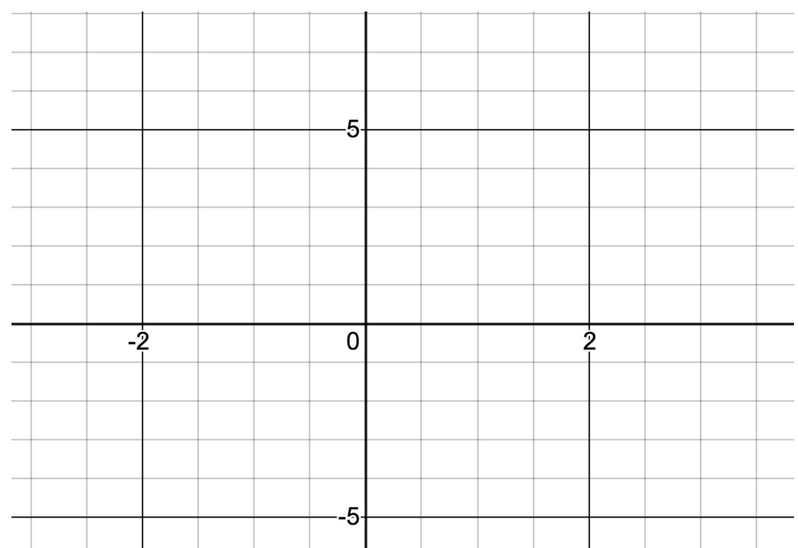
Local Extrema

What can we learn about the graph from $f''(x)$?

Concave UP

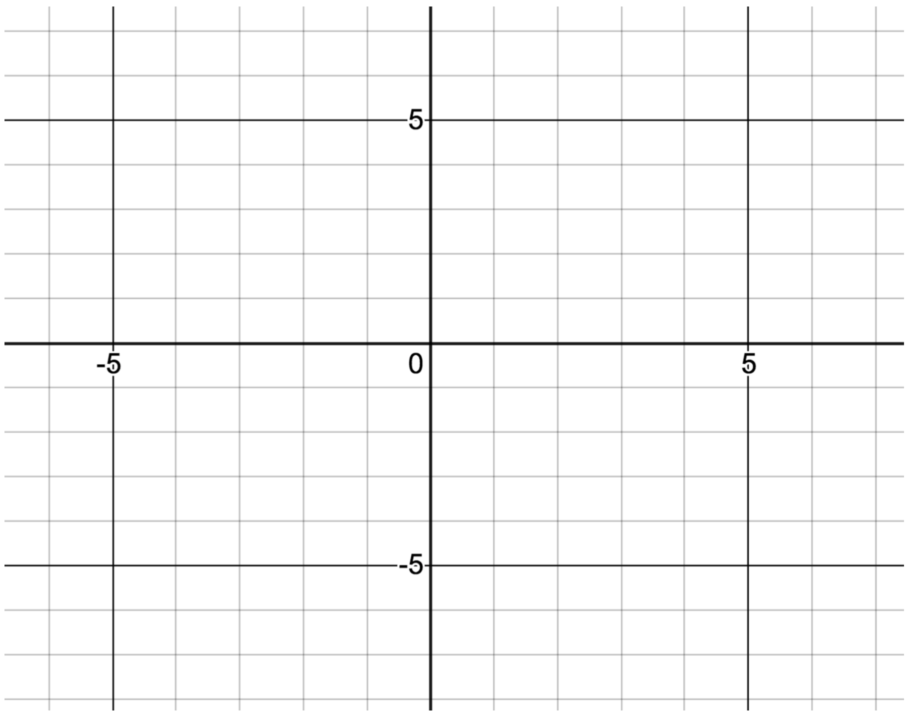
Concave DOWN

Inflection Point

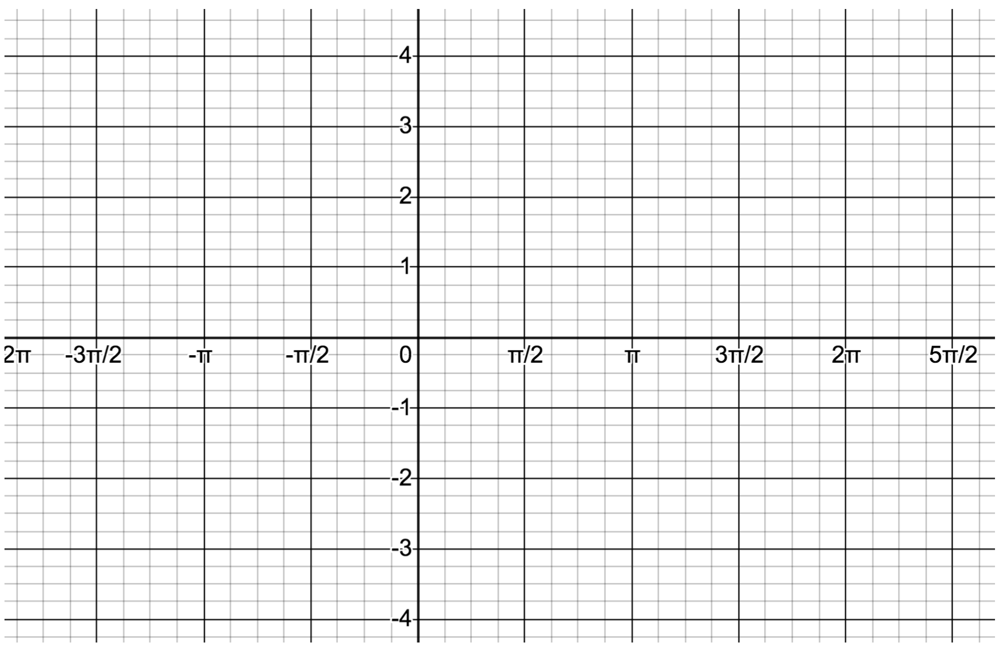


Sketch the graph of a function that satisfies the following conditions

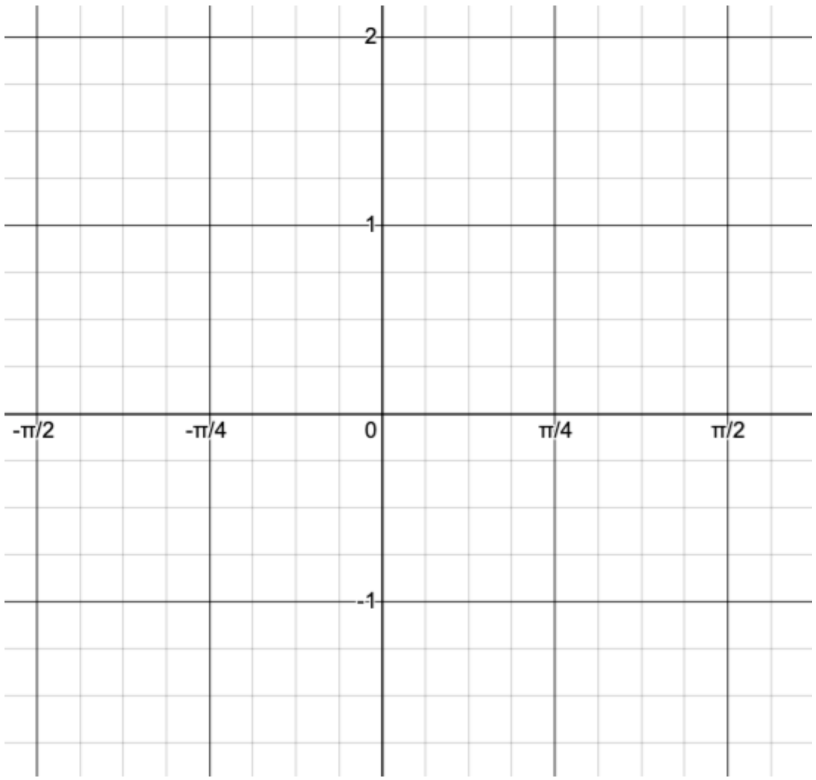
26. $f'(0) = f'(4) = 0, \quad f'(x) = 1$ if $x < -1$,
$f'(x) > 0$ if $0 < x < 2$,
$f'(x) < 0$ if $-1 < x < 0$ or $2 < x < 4$ or $x > 4$,
$\lim_{x \to 2^-} f'(x) = \infty, \quad \lim_{x \to 2^+} f'(x) = -\infty$,
$f''(x) > 0$ if $-1 < x < 2$ or $2 < x < 4$,
$f''(x) < 0$ if $x > 4$



Example: Graph and discuss $f(x)=\cos^2(x)-2\sin(x)$



Example: Graph and discuss $f(x) = 2x - \tan x \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$



3.5 More Graphing – Asymptotes.

Vertical Asymptote:

$$\text{“}\ \frac{\textit{nonzero}}{0}\ \text{“}$$

End Behavior / Horizontal Asymptote: $\lim\limits_{x\to} f(x) =$

Slant Asymptotes: $\lim\limits_{x\to} f(x) =$, but in a predictable way.

Example: $f(x) = \dfrac{x^2}{x-1}$

$f(x) = 3x + e^{-x}$ (using intuition since exponentials in 5B)

Note: There are also parabolic asymptotes, etc.

Example: Sketch and discuss $f(x) = \dfrac{x^2}{x-1}$

What can we learn about the graph from $f(x)$?

Domain

Behavior near domain issues/Discontinuities/Vertical Asymptotes

X intercepts

Y intercepts

End behavior

What can we learn about the graph from $f'(x)$?

Horizontal Tangents/Vertical Tangents/Cusps

Increasing

Decreasing

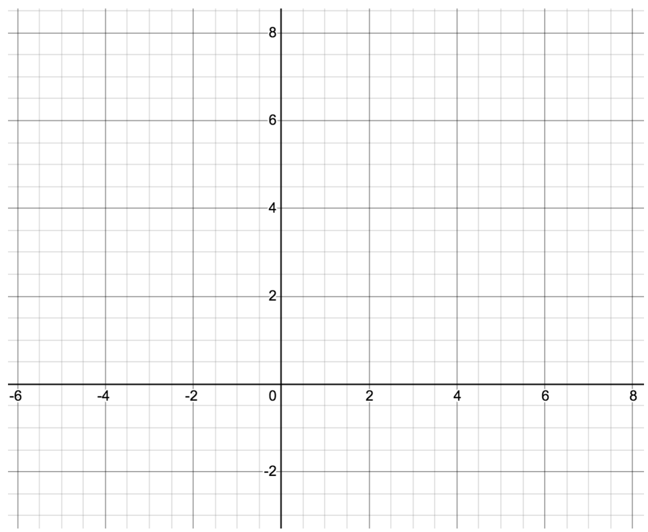
Local Extrema

What can we learn about the graph from $f''(x)$?

Concave UP

Concave DOWN

Inflection Point



Example: Sketch and discuss $f(x)=\sqrt{\dfrac{x}{x-5}}$

What can we learn about the graph from $f(x)$?

Domain

Behavior near domain issues/Discontinuities/Vertical Asymptotes

X intercepts

Y intercepts

End behavior

What can we learn about the graph from $f'(x)$?

Horizontal Tangents/Vertical Tangents/Cusps

Increasing

Decreasing

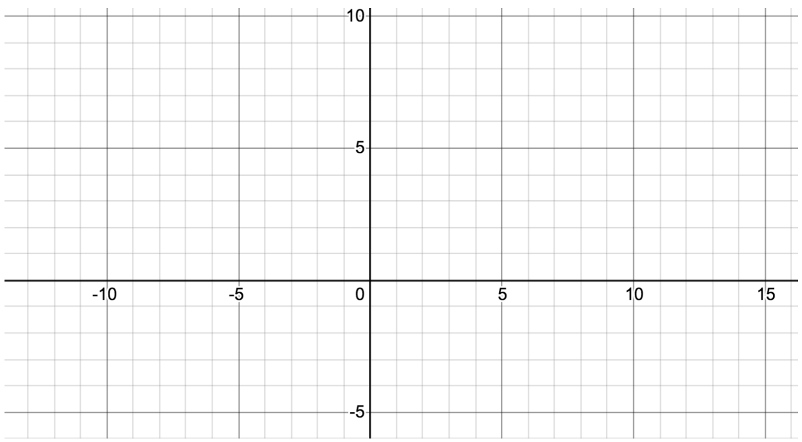
Local Extrema

What can we learn about the graph from $f''(x)$?

Concave UP

Concave DOWN

Inflection Point



3.7ii Absolute Extrema

Previously we discussed finding absolute extrema in the special case where f(x) is ______________________ on ________________

What makes that case special is ____

How do we find absolute extrema if we don't have this special case?

Example: Find the absolute extrema of $f(x) = 3x^4 - 16x^3 + 18x^2$ ~~on [-1,4]~~

Method? We have to ______________________

We start by finding critical numbers, but how do we VALIDATE that those crit. #s actually lead to absolute extremum?

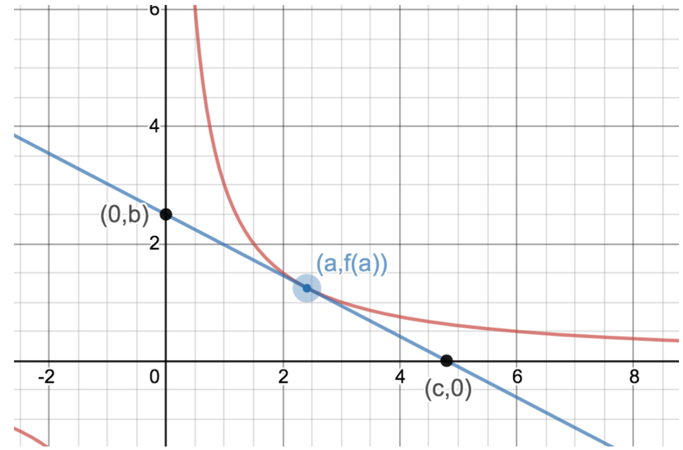
In physical problems, we can often validate existence of an absolute extremum with a physical explanation.

Example:

15. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Example: See desmos animation on 5A page: https://www.desmos.com/calculator/0ckuig0ohk

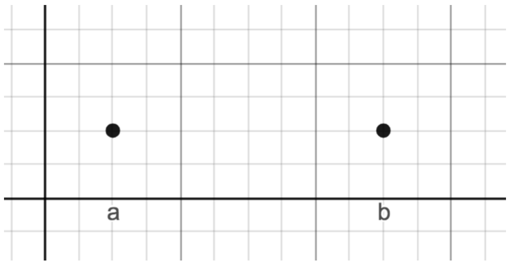
57. What is the shortest possible length of the line segment that is cut off by the first quadrant and is tangent to the curve $y = 3/x$ at some point?

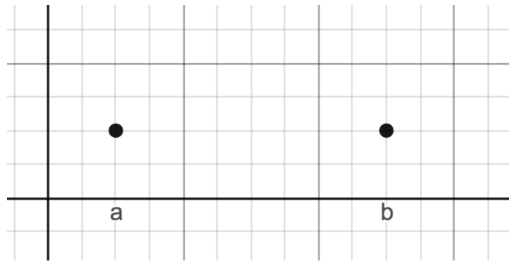


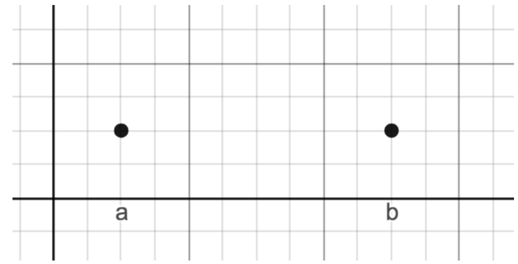
3.2 Rolles Theorem and the Mean Value Theorem

We are going to look at two very important theorems that give an algebraic relationship between $f(x)$ & $f'(x)$

Intuitively: Suppose you were asked to sketch a graph of a function defined on [a,b] with the only requirement be that $f(a)=f(b)$, $f(x)$ is continuous and differentiable.





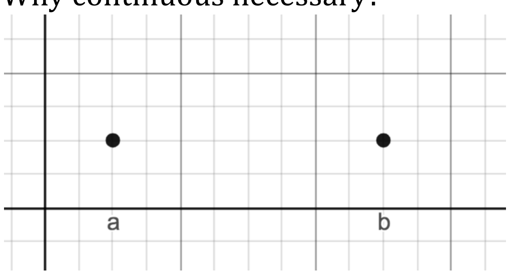


Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

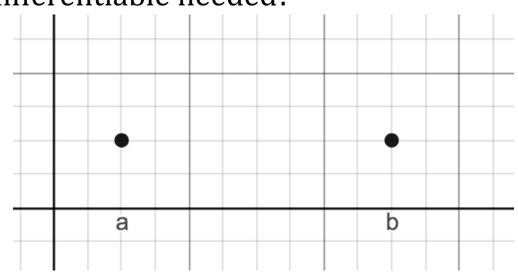
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Why continuous necessary?



Why differentiable needed?



Proof of Rolle's Theorem (different than book)

The Extreme Value Theorem tells us that if a function $f(x)$ is continuous on closed interval [a, b}, then

Case 1: Supposed m=M

Case 2: Suppose

Example: For the following two functions: Does Rolle's Theorem apply on the given interval? If so, find "c"

$f(x)=\sqrt{9-x^2}; \quad [-3,3]$

$f(x)=x^{2/3}; \quad [-1,1]$

Rolle's Theorem is often very useful in proofs.

Example 2 pg 221: "Existence and Uniqueness Proof"

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

To prove exactly one we must prove two things: at least one (Existence) and not more than one (Uniqueness)

At least one:

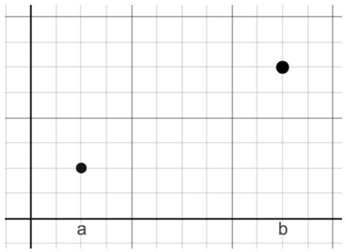
Not more than one: "Proof by Contradiction"

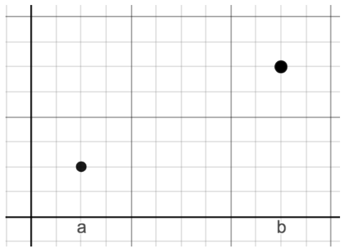
Mean Value Theorem MVT

Intuitive Idea:

(1) Suppose you average 40 mph on your way home. Are you ever going exactly 40 mph?

(2) Suppose you were asked to sketch a graph of a function defined [a,b] with the only requirement be that $f(x)$ is continuous and differentiable.





The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

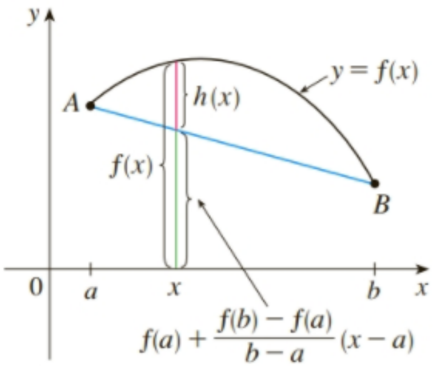
$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

<u>Proof of MVT:</u>

Let $h(x)$ be the vertical distance between $f(x)$ and the line connecting A and B as shown:



Apply Rolle's Theorem to $h(x)$ on [a,b]

Example: Does The Mean Value Theorem apply to $f(x) = \dfrac{2}{3-x}$ interval [1, 2]? If so, find "c"

The MVT is often used in proofs:

Prove that if $f'(x) > 0$ on an interval then $f(x)$ is increasing on that interval.

We've been using this as fact, but it has not been proven.

See book's proofs of Theorem 5 and Corallary 7 on page 224+.